Two-level system with a large velocity gradient

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Corrections highlighted in yellow

To test the ability of radiative transfer codes to model clouds with large velocity gradients, I suggest considering a two level case similar to that considered for the static case. Now, however, we introduce a large velocity gradient and – in the limit of large optical depth – can obtain an exact analytic solution.

As for the static case, we adopt an **excitation model** with

- Two-level ortho-water molecule (1_{01} and 1_{10} states)
- Collisional excitation of 557 GHz transition by ortho and para-H_2 in 3:1 ratio*

  ➔ de-excitation rate coefficient, $q_{21} = 2.18 \times 10^{-10}$ cm^3 s^{-1}

  excitation rate coefficient, $q_{12} = 1.12 \times 10^{-10}$ cm^3 s^{-1}

  critical density, $n_{cr} = 1.59 \times 10^{7}$ cm^{-3}

*Realistically, the ratio is almost certainly smaller than 3

and a **cloud model** with:

Outer radius, R_{out} = 0.1 pc
Inner radius = 0.001 pc
H_2 density = 10^4 cm^{-3}
Temperature = 40 K
No external radiation

Now, however, we introduce a large radial outflow velocity

$v_R = \alpha R$, with α = 100 km/s/pc

and adopt a Gaussian line width $\Delta v \ll v_R(R_{out})$ = 10 km/s

The Sobolev optical depth is given by

$$\tau_S = (f_1 - f_2)\, A_{21}\, \lambda^2\, n(\text{o–H}_2\text{O}) / [8\,\pi\,\alpha]$$

$$= (1 - 2f_2)\, t \qquad (1)$$

where f_l and $f_2 = 1 - f_l$ are the populations in the lower (1_{01}) and upper (1_{10}) states (as a fraction of the total ortho–H_2O abundance),

and where $\mathbf{t = x(o–H_2O) / 8.132 \times 10^{-9}}$ and x(o–H_2O) is the abundance of ortho-water The quantity t represents the Sobolev optical depth that would have resulted had all the water molecules been in the lower state.

Level populations

Except within a thin shell close to the surface (with a fractional thickness $\Delta v / v_R[R_{out}]$), the escape probability approximation is an excellent one, and the level populations are ***independent of location*** and given by the equation of statistical equilibrium

$$\mathbf{f_2/f_l = q_{12} / (q_{21} + A_{21}\beta) = (q_{12}/q_{21}) / (1 + s\beta)} \qquad \mathbf{(2)}$$

where $\beta = (1 - \exp [-\tau_S]) / \tau_S$

and $s = n_{cr} / n[H_2] = 1588$

In the limit $\tau_S << 1$, we obtain $\beta = 1$

➔ $f_2 = (q_{12}/q_{21}) / (1 + s) = 3.23 \times 10^{-4}$

(as in the static, optically-thin case)

In the limit $\tau_S >$ few, we may write $\beta \sim 1/\tau_S$

➔ $$\mathbf{f_2 / (1 - f_2) = f_2/f_1 = (q_{12}/q_{21}) / (1 + s/\tau_S) = \tau_S\, u / (\tau_S + s)} \qquad \mathbf{(3)}$$

where $\mathbf{u = (q_{12}/q_{21}) = 0.512}$

Solving this equation simultaneously with equation (1), we obtain a quadratic equation for f_2:

$f_2 / (1 - f_2) = u\,t\,(1 - 2f_2) / [(1 - 2f_2) + s]$

➔ $2(u + 1)\, t\, f_2^2 - (3[u+1]t + s)\, f_2 + ut = 0$

The solution is

$$\mathbf{f_2 = \{(3u + 1)t + s - [(1 - u)^2\, t^2 + 2(3u + 1)\, st + s^2\,]^{1/2} \} / \{4(u+1)t\}}$$

$$\mathbf{= \{2.536\, t + 1588 - [0.238144\, t^2 + 8054.336\, t + 2521744\,]^{1/2} \} / \{6.048\, t\}} \qquad \mathbf{(4)}$$

which has the limiting behaviors

$f_2 = u / (u + 1) = 0.339$ (LTE value) in the limit $t >> s$ (large abundance)

and

$f_2 = ut/s = 3.23 \times 10^{-4}\, t$ in the regime few $> t >> s$ (intermediate abundance)

NOTE: The above expressions only applies in the limit $t >$ few, because we have used the approximation $\beta = (1 - \exp[-\tau_S])/\tau_S \sim 1/\tau_S$

However, for small t, and in the limit $s >> 1$, we may approximate $t \sim \tau_S$ and write

$$\mathbf{f_2 = u/[s\beta] = u\, t / [s\,(1 - e^{-t})] = 3.23 \times 10^{-4}\, t / (1 - e^{-t})} \qquad \mathbf{(5)}$$

In Figure 1, the 1_{10} level population f_2 is plotted as a function of the ortho-water abundance, using equation (5) for abundances less than 3×10^{-8} (corresponding to $t < 3.7$), and equation (4) for greater abundances.

Line luminosity

The total line luminosity is

$$\mathbf{L = \mathcal{N}(o\text{–}H_2O)\, A_{21}\, h\, \nu\, f_2\, \beta \;=\; 1.57 \times 10^{40}\, f_2\, \beta\, x(o\text{–}H_2O)\ erg\ s^{-1}} \qquad \mathbf{(6)}$$

where $\mathcal{N}(o\text{–}H_2O) = 1.23 \times 10^{57}\, x(o\text{–}H_2O)$ is the number of ortho-water molecules in the cloud.

The result is plotted in Figure 2.

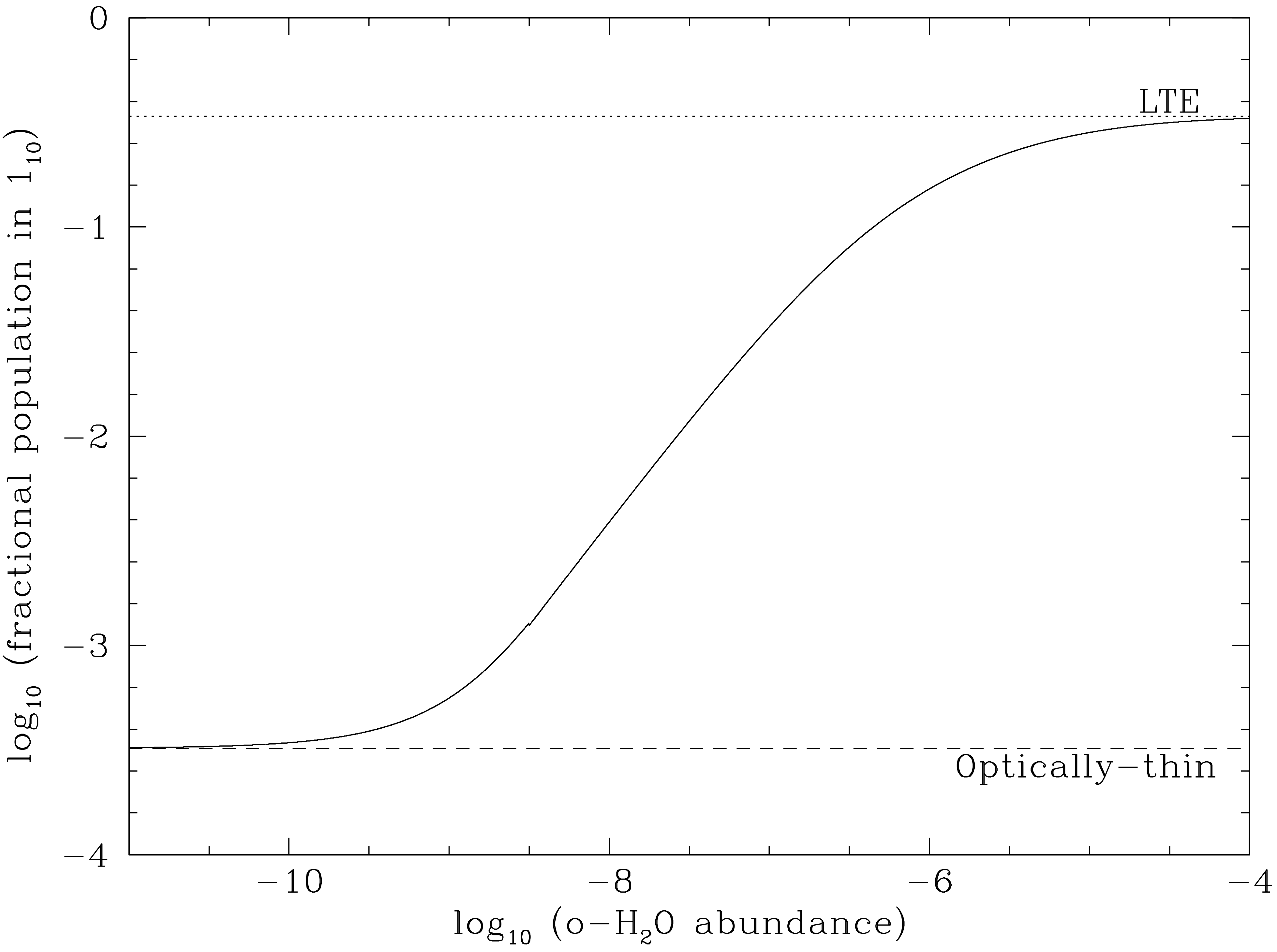
In the limit $t << s$, we may write $f_2\, \beta = u/s = 3.23 \times 10^{-4}$ and the luminosity is

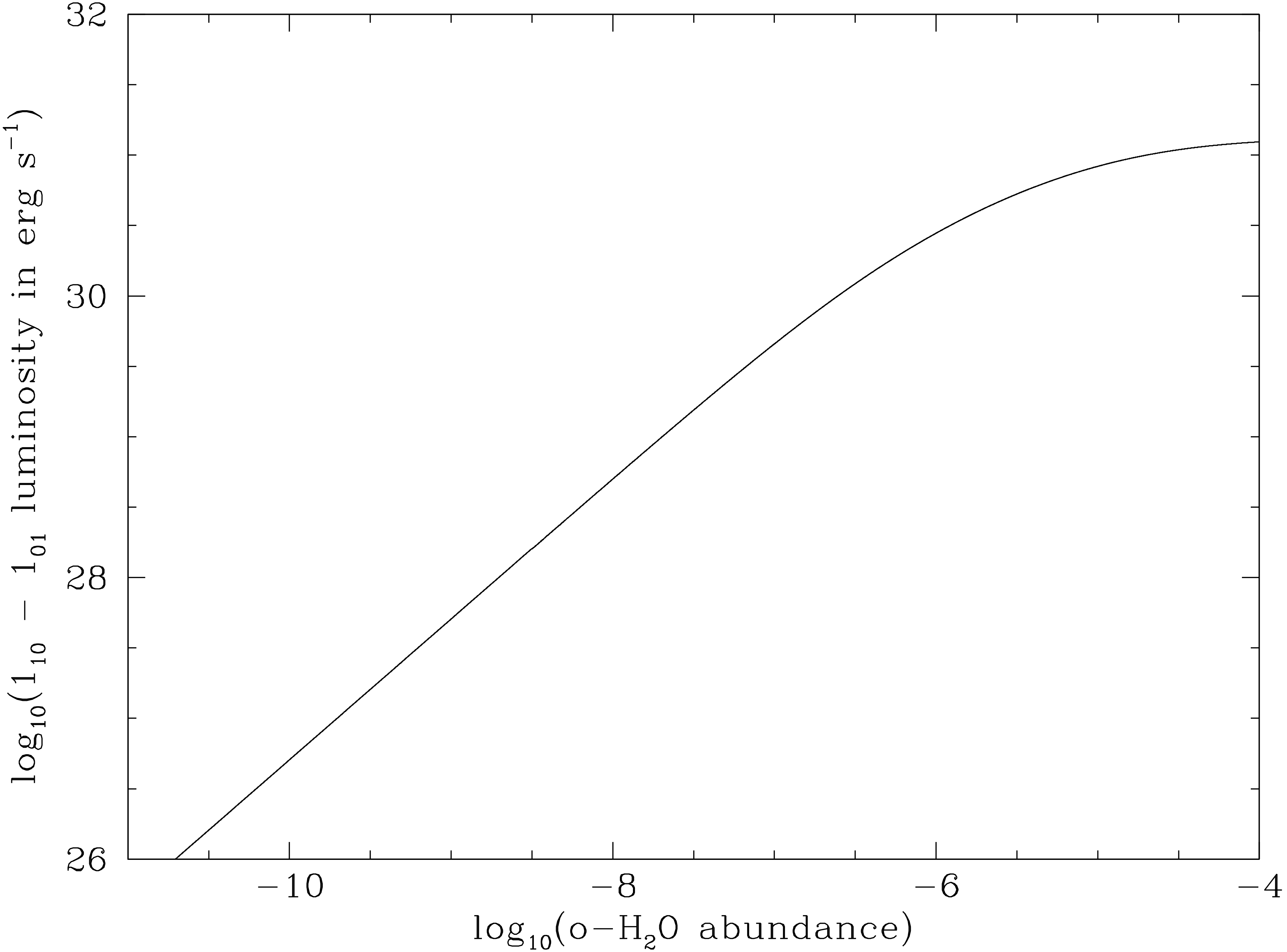
$$L = 5.08 \times 10^{26}\, [x(o\text{–}H_2O)/10^{-10}]\ \mathrm{erg\ s^{-1}} = 4.13 \times 10^{27}\ t\ \mathrm{erg\ s^{-1}}$$

(in agreement with the result previously obtained for the static case).

In the limit $t >> s$, the gas is thermalized and optically-thick.

Under these conditions, $f_2 = 0.339$ and $\beta = 1 / [(1 - 2f_2)\, t] = 3.11 / t$
which implies $L = 1.35 \times 10^{31}$ erg s^{-1}. This is the maximum luminosity.


0
−1
−2
−3
−4
−10
−8
−6
−4
log$_{10}$ (fractional population in 1$_{10}$)
log$_{10}$ (o−H$_2$O abundance)
LTE
Optically−thin


32
30
28
26
−10
−8
−6
−4
$\log_{10}(1_{10} - 1_{01}$ luminosity in erg $s^{-1})$
$\log_{10}$(o−H_2O abundance)